

## MODULAR CATEGORIES AND ORBIFOLD MODELS

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ABSTRACT. In this paper, we try to answer the following question: given a modular tensor category  $\mathcal{A}$  with an action of a compact group  $G$ , is it possible to describe in a suitable sense the “quotient” category  $\mathcal{A}/G$ ? We give a full answer in the case when  $\mathcal{A} = \mathcal{V}ec$  is the category of vector spaces; in this case,  $\mathcal{V}ec/G$  turns out to be the category of representation of Drinfeld’s double  $D(G)$ . This should be considered as category theory analog of topological identity  $\{pt\}/G = BG$ .

This implies a conjecture of Dijkgraaf, Vafa, E. Verlinde and H. Verlinde regarding so-called orbifold conformal field theories: if  $\mathcal{V}$  is a vertex operator algebra which has a unique irreducible module,  $\mathcal{V}$  itself, and  $G$  is a compact group of automorphisms of  $\mathcal{V}$ , and some not too restrictive technical conditions are satisfied, then  $G$  is finite, and the category of representations of the algebra of invariants,  $\mathcal{V}^G$ , is equivalent as a tensor category to the category of representations of Drinfeld’s double  $D(G)$ . We also get some partial results in the non-holomorphic case, i.e. when  $\mathcal{V}$  has more than one simple module.

## INTRODUCTION

The goal of this paper is to discuss the properties of the so-called orbifold models of Conformal Field Theory from the categorical point of view.

For readers convenience, we recall here the main definitions and results, assuming that the reader is familiar with the notion of a vertex operator algebra. Let  $\mathcal{V}$  be a VOA and  $G$  — a finite group acting on  $\mathcal{V}$  by automorphisms. Then the subspace of invariants  $\mathcal{V}^G$  is itself a VOA. The main question is: is it possible to describe the category of  $\mathcal{V}^G$ -modules in terms of the category of  $\mathcal{V}$ -modules and the group  $G$ ?

This question was asked in this form in [DVVV]. They didn’t give a full answer, but did suggest (without a proof) an answer in a special case, when  $\mathcal{V}$  is holomorphic, i.e. has only one simple module (vacuum module). In this case, the category of representations of  $\mathcal{V}$  is equivalent to the category of vector spaces. The answer suggested in [DVVV] and further discussed in [DPR] is that in this case, the category of  $\mathcal{V}^G$ -modules is equivalent to the category of modules over the (twisted) Drinfeld double  $D(G)$  of the group  $G$ . The case of VOA’s coming from Wess-Zumino-Witten model (or, equivalently, from affine Lie algebras) was studied in detail in [KT].

Many of the results of [DVVV] were rigorously proved in the language of VOA’s in a series of papers of Dong, Li, and Mason; in particular, it is proved in [DLM1] that in the holomorphic case,  $\mathcal{V}$  considered as a module over both  $\mathcal{V}^G$  and  $G$  can be written as

$$(0.1) \quad \mathcal{V} = \bigoplus_{\lambda \in \hat{G}} V_{\lambda} \otimes M_{\lambda}$$

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where  $V_\lambda$  are irreducible  $G$ -modules and  $M_\lambda$  are non-zero simple pairwise non-isomorphic  $\mathcal{V}^G$ -modules. However, even in the holomorphic case the full result (i.e., that the category of  $\mathcal{V}^G$ -modules is equivalent to the category of modules over the (twisted) Drinfeld double  $D(G)$ ) is still not proved.

In this paper we suggest a new approach to the problem. The main idea of this approach is **not** using the structure theory of VOA's. In the author's opinion, all the information about VOA's which is relevant for this problem is encoded in the category of representations of  $\mathcal{V}$ . For example, the pair  $\mathcal{V}^G \subset \mathcal{V}$  can be described in this way: as discussed in [KO], such a pair is the same as an associative commutative algebra in the category  $\mathcal{C} = \text{Rep } \mathcal{V}^G$  with some technical restrictions. Thus, if we know some basic properties of  $\mathcal{C}$  — e.g., that this category is semisimple, braided, and rigid — then we can forget anything else about VOA's, operator product expansions, etc. Instead, we use well-known tools for working with braided tensor categories, such as graphical presentation of morphisms.

Using this approach, in this paper we give an accurate proof of the above conjecture; for simplicity, we only consider the case when all the “twists”, i.e. phase factors, are trivial. In this case, the main result reads as follows.

**Theorem.** *Let  $\mathcal{V}$  be a VOA,  $G$  — a finite group of automorphisms of  $\mathcal{V}$ , and  $\mathcal{V}^G$  — the algebra of invariants. Assume that*

- (1)  $\mathcal{V}$  is “holomorphic”, i.e. has a unique irreducible module,  $\mathcal{V}$  itself, so that  $\text{Rep } \mathcal{V} = \text{Vec}$ .
- (2)  $\text{Rep } \mathcal{V}^G$  is a semisimple braided rigid balanced category
- (3)  $\mathcal{V}$  has finite length as a  $\mathcal{V}^G$ -module
- (4) Certain cohomology class  $\omega \in H^3(G, \mathbb{C}^\times)$  defined by  $\mathcal{V}$  is trivial

*Then the category  $\text{Rep } \mathcal{V}^G$  is equivalent to the category of modules over  $D(G, H) = \mathbb{C}[G] \ltimes \mathcal{F}(H)$  for some normal subgroup  $H \subset G$ . If, in addition, we assume that  $\text{Rep } \mathcal{V}^G$  is modular, then  $H = G$  so  $\text{Rep } \mathcal{V}^G \simeq \text{Rep } D(G)$ .*

In this paper we assume that the reader is well familiar with braided tensor categories and in particular, with the technique of using graphs to prove identities in such categories, developed by Reshetikhin and Turaev. This can be found in many textbooks (see, e.g., [Ka]); we follow the conventions of [BK]. Conversely, knowledge of vertex operator algebras and conformal field theory is not required: they do not even appear in the paper except in this introduction.

The paper also makes heavy use of results of [KO], so we suggest that the reader keep a copy of that paper handy.

## 1. PRELIMINARIES

Throughout the paper, we denote by  $\mathcal{C}$  a semisimple braided tensor category over  $\mathbb{C}$ , with simple objects  $L_i, i \in I$  (“simple” always means “non-zero simple”). As usual, we assume that the unit object is simple and denote the corresponding index in  $I$  by 0:  $\mathbf{1} = L_0$ . We assume that all spaces of morphisms are finite-dimensional and denote  $\langle V, W \rangle = \dim \text{Hom}_{\mathcal{C}}(V, W)$ ; in particular,  $\langle L_i, V \rangle$  is the multiplicity of  $L_i$  in  $V$ .

As any abelian category,  $\mathcal{C}$  is a module over the category  $\text{Vec}$  of finite-dimensional complex vector spaces, i.e. we have a natural functor of “external tensor product”:

$$(1.1) \quad \boxtimes: \text{Vec} \times \mathcal{C} \rightarrow \mathcal{C}$$

defined by  $\text{Hom}(L_i, V \boxtimes L) = V \otimes \text{Hom}(L_i, L)$  (more formally:  $V \boxtimes L$  is the object representing the functor  $F(M) = V \otimes \text{Hom}(M, L)$ ). This functor is bilinear and has natural associativity properties:

$$(1.2) \quad \begin{aligned} V \boxtimes (W \boxtimes L) &= (V \otimes W) \boxtimes L \\ V \boxtimes (L \otimes M) &= (V \boxtimes L) \otimes M \\ (V \boxtimes L) \otimes (W \boxtimes M) &= (V \otimes W) \boxtimes (L \otimes M) \end{aligned}$$

(here  $=$  means “canonically isomorphic”). Abusing the language, we will sometimes use  $\otimes$  instead of  $\boxtimes$ .

Also, we denote by  $G$  a compact group (e.g.,  $G$  can be finite) and by  $\text{Rep } G$  the category of finite-dimensional complex representations of  $G$ . This category is semisimple. We denote by  $\hat{G}$  the set of isomorphism classes of simple  $G$ -modules, and for each  $\lambda \in \hat{G}$  we choose a representative  $V_\lambda$ .

We denote by  $\mathcal{C}[G]$  the category whose objects are pairs  $(M \in \mathcal{C}, \text{action of } G \text{ by automorphisms on } M)$ . In particular, each object of  $\mathcal{C}$  can be considered as an object in  $\mathcal{C}[G]$  by letting  $G$  act trivially. This category has the following properties, proof of which is left as an exercise to the reader.

- (1)  $\mathcal{C}[G]$  is a semisimple rigid braided tensor category, with simple objects  $V_\lambda \boxtimes L_i$ .
- (2) Define the functor of  $G$ -invariants  $\mathcal{C}[G] \rightarrow \mathcal{C}$  by

$$\text{Hom}_{\mathcal{C}}(L_i, M^G) = (\text{Hom}_{\mathcal{C}}(L_i, M))^G = \text{Hom}_{\mathcal{C}[G]}(L_i, M),$$

i.e.  $(V \boxtimes L_i)^G = V^G \boxtimes L_i$ . Then this functor is exact, and one has canonical embedding

$$(1.3) \quad X^G \otimes Y^G \hookrightarrow (X \otimes Y)^G$$

- (3) One can define canonical functor of “exterior tensor product”

$$\boxtimes: \text{Rep } G \times \mathcal{C}[G] \rightarrow \mathcal{C}[G]$$

which has the associativity properties (1.2).

## 2. UNTWISTED SECTOR

Throughout the paper, we let  $A$  be a  $\mathcal{C}$ -algebra (i.e. an object of  $\mathcal{C}$  with a map  $\mu: A \otimes A \rightarrow A$ ) as defined in [KO]. We also assume that  $G$  acts on  $A$  by multiplication-preserving automorphisms and that  $A^G = \mathbf{1}$  (recall that by axioms of  $\mathcal{C}$ -algebra, one has canonical embedding  $\mathbf{1} \hookrightarrow A$  and the multiplicity  $\langle A, \mathbf{1} \rangle = 1$ ). In addition, we will also assume that  $\mathcal{C}$  is rigid and balanced, and that  $A$  is rigid and satisfies  $\theta_A = \text{id}$ .

Our main goal is to describe the category  $\mathcal{C}$  in terms of the group  $G$  and the category  $\text{Rep}^0 A$  (see [KO] for definitions). The main motivation for this comes from the orbifold conformal field theories, as explained in the introduction; in this situation,  $\mathcal{C} = \text{Rep } \mathcal{V}^G$ , and  $A$  is  $\mathcal{V}$  considered as a  $\mathcal{V}^G$ -module (cf. [KO]). We will freely use results and notation of [KO].

We start by describing the structure of  $A$  as an object of  $\mathcal{C}$ .

Define a functor  $\Phi: \text{Rep } G \rightarrow \mathcal{C}$  by

$$(2.1) \quad V \mapsto (V \boxtimes A)^G.$$

In other words, if one writes

$$A = \bigoplus_{\lambda \in \widehat{G}} V_\lambda \boxtimes M_\lambda$$

for some  $M_\lambda \in \mathcal{C}$ , then  $\Phi(V_\lambda^*) = M_\lambda$ .

**2.1. Theorem.**  $\Phi(\mathbb{C}) = \mathbf{1}$  and  $\langle \Phi(V_\lambda), \mathbf{1} \rangle = 0$  for  $V_\lambda \not\cong \mathbb{C}$ .

*Proof.* Immediate from definitions.  $\square$

Our next goal is to prove that under suitable conditions,  $\Phi$  is a tensor functor. An impatient reader can find the final result as Theorem 2.11 below. We start by constructing a morphism  $\Phi(V) \otimes \Phi(W) \rightarrow \Phi(V \otimes W)$ .

**2.2. Theorem.** *Define the functorial morphism  $J: \Phi(V) \otimes \Phi(W) \rightarrow \Phi(V \otimes W)$  as the following composition:*

$$(V \boxtimes A)^G \otimes (W \boxtimes A)^G \hookrightarrow ((V \otimes W) \boxtimes (A \otimes A))^G \xrightarrow{\mu} ((V \otimes W) \boxtimes A)^G$$

(we have used (1.2), (1.3)). Then  $J$  is compatible with associativity, commutativity, unit, and balancing morphisms in  $\text{Rep } G, \mathcal{C}$ .

*Proof.* Immediate from definitions.  $\square$

This theorem allows one to define functorial morphisms  $J: \Phi(W_1) \otimes \dots \otimes \Phi(W_n) \rightarrow \Phi(W_1 \otimes \dots \otimes W_n)$ . For future use, we explicitly write the functoriality property: for any  $f: W_i \rightarrow W'_i$ , the following diagram is commutative

$$\begin{array}{ccc} \Phi(W_1) \otimes \dots \otimes \Phi(W_n) & \xrightarrow{1 \otimes \dots \otimes \Phi(f) \otimes \dots \otimes 1} & \Phi(W_1) \otimes \dots \otimes \Phi(W'_i) \otimes \dots \otimes \Phi(W_n) \\ J \downarrow & & J \downarrow \\ \Phi(W_1 \otimes \dots \otimes W_n) & \xrightarrow{\Phi(1 \otimes \dots \otimes f \otimes \dots \otimes 1)} & \Phi(W_1 \otimes \dots \otimes W'_i \otimes \dots \otimes W_n) \end{array}$$

**2.3. Remark.** We do not claim that  $J$  is an isomorphism: in general, this is false.

Now let us use rigidity of  $A$ . We denote by  $e_V: V^* \otimes V \rightarrow \mathbf{1}, i_V: \mathbf{1} \rightarrow V \otimes V^*$  the canonical rigidity morphisms, and by  $\dim M$  dimension of an object  $M \in \mathcal{C}$ .

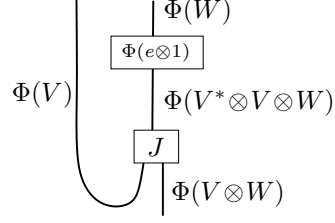
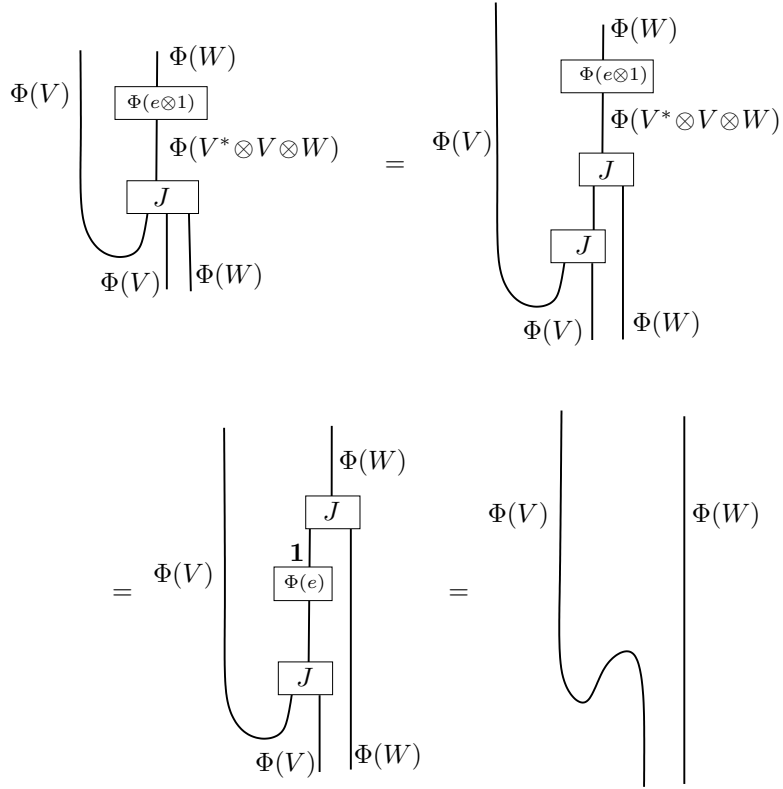
**2.4. Theorem.** *For every  $V \in \text{Rep } G$ , the map*

$$(2.2) \quad \tilde{e}: \Phi(V^*) \otimes \Phi(V) \xrightarrow{J} \Phi(V^* \otimes V) \xrightarrow{\Phi(e)} \Phi(\mathbb{C}) = \mathbf{1}$$

*gives an isomorphism  $\Phi(V^*) \simeq \Phi(V)^*$ . In other words, there exists a morphism  $\tilde{i}: \mathbf{1} \rightarrow \Phi(V) \otimes \Phi(V^*)$  such that  $\tilde{e}, \tilde{i}$  satisfy the rigidity axioms.*

*Proof.* It is easy to see that for any  $X \in \mathcal{C}[G]$ , the morphism  $(X^*)^G \otimes X^G \hookrightarrow (X^* \otimes X)^G \rightarrow \mathbf{1}^G = \mathbf{1}$  identifies  $(X^*)^G \simeq (X^G)^*$  (here we have used (1.3) and functoriality of  $X \mapsto X^G$ ). Combining this with the definition of  $J$  and rigidity of  $A$  we get the statement of the theorem.  $\square$

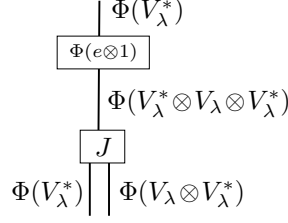
**2.5. Theorem.** *For any  $V, W \in \text{Rep } G$ , the morphism  $J: \Phi(V) \otimes \Phi(W) \rightarrow \Phi(V \otimes W)$  is injective.*

FIGURE 1. Definition of  $I$ FIGURE 2. Proof of  $IJ = \text{id}$ 

*Proof.* Define the morphism  $I: \Phi(V \otimes W) \rightarrow \Phi(V) \otimes \Phi(W)$  by the graph shown in Figure 1.

Then the composition  $IJ: \Phi(V) \otimes \Phi(W) \rightarrow \Phi(V) \otimes \Phi(W)$  can be rewritten as shown in Figure 2 and thus,  $IJ = \text{id}$  which proves that  $J$  is injective.  $\square$

**2.6. Theorem.** *Let  $V_\lambda$  be an irreducible representation of  $G$ .*

FIGURE 3. Definition of  $f: \Phi(V_\lambda^*) \otimes \Phi(V_\lambda \otimes V_\lambda^*) \rightarrow \Phi(V_\lambda^*)$ 

- (1)  $\Phi(V_\lambda)$  is either zero or a simple object in  $\mathcal{C}$ .
- (2) If  $\lambda, \mu \in \widehat{G}$ ,  $\lambda \neq \mu$  are such that  $\Phi(V_\lambda) \neq 0$ ,  $\Phi(V_\mu) \neq 0$ , then  $\Phi(V_\lambda) \not\cong \Phi(V_\mu)$ .

*Proof.* Using Theorem 2.1, Theorem 2.5, we get

$$\langle \Phi(V_\lambda) \otimes \Phi(V_\mu^*), \mathbf{1} \rangle \leq \langle \Phi(V_\lambda \otimes V_\mu^*), \mathbf{1} \rangle = \delta_{\lambda\mu}$$

Thus, using  $\Phi(V^*) \simeq \Phi(V)^*$  (see Theorem 2.4) for  $\lambda = \mu$  we get  $\langle \Phi(V_\lambda) \otimes \Phi(V_\lambda)^*, \mathbf{1} \rangle \leq 1$ , which shows that  $\Phi(V_\lambda)$  is either simple or zero, and for  $\lambda \neq \mu$ , we get  $\langle \Phi(V_\lambda) \otimes \Phi(V_\mu)^*, \mathbf{1} \rangle = 0$ .  $\square$

**2.7. Lemma.** Let  $V_\lambda$  be an irreducible representation of  $G$  such that  $\Phi(V_\lambda) \neq 0$ . Then the composition

$$\mathbf{1} \xrightarrow{\tilde{i}} \Phi(V_\lambda) \otimes \Phi(V_\lambda^*) \xrightarrow{J} \Phi(V_\lambda \otimes V_\lambda^*)$$

is equal to  $\Phi(i_\lambda)$ , where  $i_\lambda$  is the canonical map  $\mathbb{C} \rightarrow V_\lambda \otimes V_\lambda^*$ .

**2.8. Remark.** For  $\Phi(V_\lambda) = 0$ , this lemma obviously fails.

*Proof.* Denote this composition by  $\varphi$ . Since  $\langle \Phi(V_\lambda \otimes V_\lambda^*), \mathbf{1} \rangle = 1$ , one has  $\varphi = c\Phi(i_\lambda)$  for some  $c \in \mathbb{C}$ . To find  $c$ , compose both  $\varphi$  and  $\Phi(i_\lambda)$  with the morphism  $f: \Phi(V_\lambda^*) \otimes \Phi(V_\lambda \otimes V_\lambda^*) \rightarrow \Phi(V_\lambda^*)$  shown in Figure 3. Arguing as in the proof of Theorem 2.5, we see that  $f\varphi = f\Phi(i_\lambda) = \text{id}_{\Phi(V_\lambda^*)}$ .  $\square$

**2.9. Theorem.** If  $\lambda \in \widehat{G}$  such that  $\Phi(V_\lambda) \neq 0$ , then  $J: \Phi(V_\lambda) \otimes \Phi(W) \rightarrow \Phi(V_\lambda \otimes W)$  is an isomorphism for any  $W \in \text{Rep } G$ .

*Proof.* Let  $I$  be as in the proof of Theorem 2.5. Let us calculate  $J I$ . Using Lemma 2.7 and functoriality of  $J$ , we can rewrite  $J I$  as shown in Figure 4.

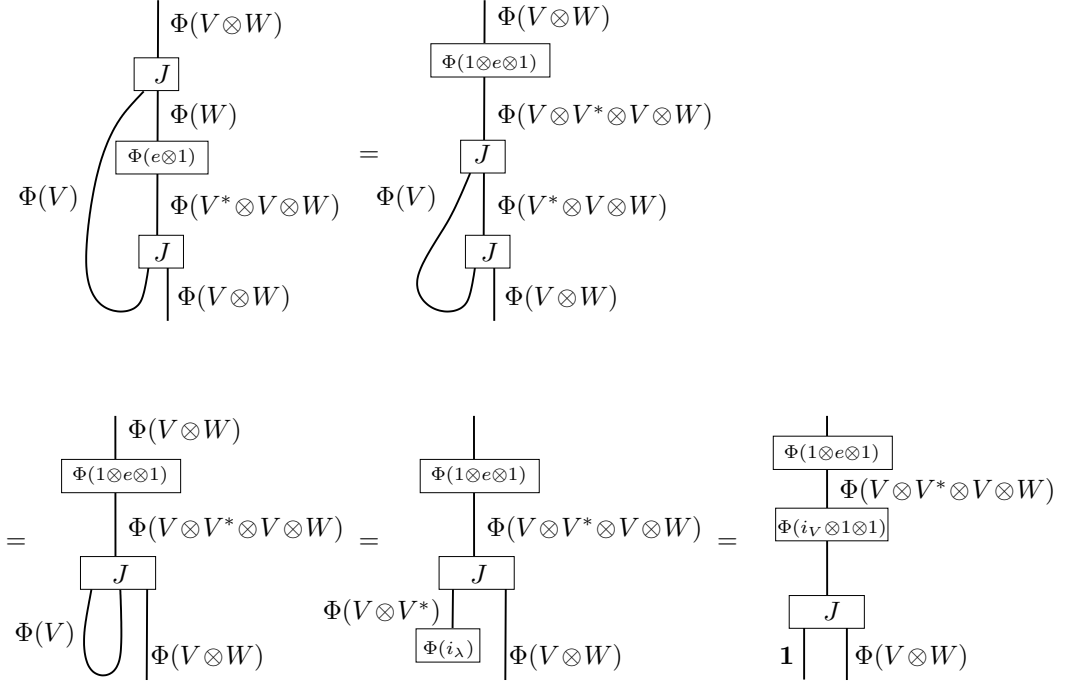
Thus,  $J I = \text{id}$ . On the other hand, it was proved in Theorem 2.5 that  $I J = \text{id}$ .  $\square$

Now, let us assume that the action of  $G$  on  $A$  is faithful, that is, every  $g \in G$ ,  $g \neq 1$  acts on  $A$  by a non-trivial automorphism.

**2.10. Theorem.** If the action of  $G$  is faithful, then  $\Phi(V_\lambda) \neq 0$  for any  $\lambda \in \widehat{G}$ .

*Proof.* Let  $I = \{\lambda \in \widehat{G} \mid \Phi(V_\lambda) \neq 0\}$ . Then, by Theorem 2.4,  $I$  is closed under duality (as usual, we denote by  $\lambda^*$  the class of representation  $V_\lambda^*$ ). Denote by  $N_{\lambda\mu}^\nu$  the multiplicities in the tensor product decomposition:  $V_\lambda \otimes V_\mu \simeq \sum N_{\lambda\mu}^\nu V_\nu$ . Then

$$(2.3) \quad N_{\lambda\mu}^\nu = 0 \quad \text{if } \lambda, \mu \in I, \nu \notin I.$$

FIGURE 4. Proof of  $J I = \text{id}$ 

Indeed: if  $N_{\lambda\mu}^\nu \neq 0$ , then there is an embedding  $V_\nu \subset V_\lambda \otimes V_\mu$ , which gives

$$\Phi(V_\nu) \subset \Phi(V_\lambda \otimes V_\mu) \simeq \Phi(V_\lambda) \otimes \Phi(V_\mu) \simeq 0$$

(by Theorem 2.9), which contradicts  $\Phi(V_\nu) \neq 0$ .

By using  $N_{\lambda\mu}^\nu = N_{\lambda\nu}^{\mu*}$ , we can rewrite (2.3) as

$$(2.4) \quad N_{\lambda\mu}^\nu = 0 \quad \text{if } \lambda, \mu \in I, \nu \notin I.$$

Let  $\mathcal{A}$  be the full subcategory in  $\text{Rep } G$  generated (as an abelian category) by  $V_\lambda, \lambda \in I$ . Then it follows from (2.4) that  $\mathcal{A}$  is closed under tensor product; it is also closed under duality (by Theorem 2.4) and contains  $\mathbb{C}$  (by Theorem 2.1). By the usual reconstruction theorems, this means that  $\mathcal{A}$  is the category of representations of some group  $H$  which is a quotient of  $G$ . But by assumption, the action of  $G$  on  $A$  is faithful, which means that the action of  $G$  on  $\bigoplus_{\lambda \in I} V_\lambda$  is faithful. Thus,  $H = G$ ,  $I = \widehat{G}$ .  $\square$

Combining all the results above, we get the main theorem of this section.

**2.11. Theorem.** *Let  $A$  be a rigid  $\mathcal{C}$ -algebra,  $\theta_A = 1$ , and  $G$  — a compact group acting faithfully on  $A$ . Then*

(1)

$$(2.5) \quad A \simeq \bigoplus_{\lambda \in \widehat{G}} V_\lambda \boxtimes M_\lambda$$

- where  $M_\lambda = \Phi(V_\lambda^*)$  are non-zero, simple, and  $M_\lambda \not\simeq M_\mu$  for  $\lambda \neq \mu$ .
- (2) Let  $\mathcal{C}_1$  be the full subcategory in  $\mathcal{C}$  generated as an abelian category by  $M_\lambda, \lambda \in \widehat{G}$ . Then  $\mathcal{C}_1$  is a symmetric tensor subcategory in  $\mathcal{C}$ , and the functor  $\Phi: \text{Rep } G \rightarrow \mathcal{C}_1$  defined by (2.1) is an equivalence of tensor categories.

**2.12. Corollary.**  $G$  is finite.

*Proof.* Immediate from (2.5) and finite-dimensionality of spaces of morphisms in  $\mathcal{C}$ .  $\square$

**2.13. Corollary.**  $\tilde{R}_{AA}^2 = \text{id}$ .

**2.14. Example.** Let  $G$  be a finite group,  $\mathcal{C} = \text{Rep } G$ ,  $A = \mathcal{F}(G)$  — the algebra of functions on  $G$ , with the usual (pointwise) multiplication. Formula  $g\delta_h = \delta_{gh}$  makes  $A$  a  $G$ -module and thus, an object of  $\mathcal{C}$ . It is trivial to show (see [KO]) that  $A$  is a rigid  $\mathcal{C}$ -algebra, and  $\text{Rep } A = \text{Vec}$ .

We also have another action of  $G$  on  $A$ , by  $\pi_g \delta_h = \delta_{hg^{-1}}$ . This commutes with previously defined and thus, defines an action of  $G$  by automorphisms on  $A$  considered as a  $\mathcal{C}$ -algebra. In this case, the functor  $\Phi$  is an equivalence of categories  $\text{Rep } G \simeq \mathcal{C}$ , and the decomposition (2.5) becomes the standard decomposition

$$\mathcal{F}(G) = \bigoplus_{\lambda \in \widehat{G}} V_\lambda \otimes V_\lambda^*.$$

Let us return to the general case.

**2.15. Theorem.** Under the assumptions of Theorem 2.11, consider  $A \in \mathcal{C}_1 \subset \mathcal{C}$  as an object of  $\text{Rep } G$  using equivalence  $\Phi$ . Then  $A \simeq \mathcal{F}(G)$  with multiplication, structure of  $G$ -module and action of  $G$  by automorphisms as defined in Example 2.14.

*Proof.* It is immediate from Theorem 2.11 that  $A$  lies in  $\mathcal{C}_1 \simeq \text{Rep}(G)$  and that as an object of  $\text{Rep}(G)$ ,  $A = \bigoplus_{\lambda \in \widehat{G}} V_\lambda \boxtimes V_\lambda^*$ . The structure of  $G$ -module is determined by the action on the second factor, and the action of  $G$  by automorphism is defined by the action on the first factor.

On the other hand, it is shown in [KO] that any algebra in category  $\text{Rep } G$  must be of the form  $A = \mathcal{F}(G/H)$  for some subgroup  $H$ . Combining these statements, we see that  $H = \{1\}$ ,  $A = \mathcal{F}(G)$ .  $\square$

**2.16. Corollary.**  $G$  is the group of all automorphisms of  $A$  as a  $\mathcal{C}$ -algebra.

**2.17. Corollary.**  $\dim A = |G|$ .

Of course, in general case  $\mathcal{C}$  can be (and usually is) larger than  $\text{Rep } G$ . A very important example is when  $\mathcal{C}$  is the category of representation of  $D(G)$  (the Drinfeld double of  $G$ ), and  $A = \mathcal{F}(G) \in \text{Rep } G \subset \text{Rep } D(G)$  with the same action of  $G$  as in Example 2.14. This example plays an important role in what follows; it is discussed in detail in Section 3.

**2.18. Example.** Let  $G$  be commutative. Then all its irreducible representations are one-dimensional, and it immediately follows from Theorem 2.11 that

$$A \simeq \bigoplus_{\lambda \in \widehat{G}} M_\lambda$$



where  $M_\lambda$  are non-zero, simple, pairwise non-isomorphic objects in  $\mathcal{C}$  and  $M_\lambda \otimes M_\mu \simeq M_{\lambda\mu}$ . This case is well studied in numerous papers under the name “simple currents extensions”; a review of known results can be found, e.g., in [FMS] and [FS].

### 3. EXAMPLE: $D(G)$

Let  $D(G) = \mathbb{C}[G] \ltimes \mathcal{F}(G)$  be the Drinfeld’s double of the finite group  $G$  (see, e.g., [BK]), and let  $\mathcal{C}$  be the category of finite-dimensional complex  $D(G)$ -modules. As is well known, this category is equivalent to the category of  $G$ -equivariant vector bundles on  $G$ . An object of this category is a complex vector space  $V$  with an action of  $G$  and with a  $G$ -grading:  $V = \bigoplus_{g \in G} V_g$  such that  $gV_x \subset V_{gxg^{-1}}$ , or, equivalently,  $\text{wt}(gv) = g\text{wt}(v)g^{-1}$ , where  $\text{wt}(v) \in G$  denotes weight of a homogeneous  $v \in V$ . The tensor product in  $\mathcal{C}$  is the usual tensor product, with  $\text{wt}(v \otimes w) = \text{wt}(v)\text{wt}(w)$ . The braiding in  $\mathcal{C}$  is given by  $\check{R} = PR$ , where

$$(3.1) \quad R = \sum_{g \in G} \delta_g \otimes g \in D(G) \otimes D(G).$$

In other words, if  $v, w$  are homogeneous vectors in  $V, W$  respectively, then

$$\check{R}(v \otimes w) = gw \otimes v, \quad g = \text{wt}(v).$$

It is also known that  $\text{Rep } D(G)$  is semisimple, and the set of isomorphism classes of simple objects is  $(g, \pi)/G$  where  $g \in G, \pi$  – an irreducible representation of the centralizer  $Z(g) = \{h \in G \mid hg = gh\}$ , and  $G$  acts on the set of pairs  $(g, \pi)$  by  $h(g, \pi) = (hgh^{-1}, \pi \circ h^{-1})$ . We will denote the corresponding representation of  $D(G)$  by  $V_{g, \pi}$ .

Let  $A = \mathcal{F}(G)$ ; consider it as object of  $\mathcal{C}$  by endowing it with the standard action of  $G$  (same as in Example 2.14), and by letting  $\text{wt}(v) = 1$  for all  $v \in A$ .

**3.1. Lemma.**  *$A$  is a rigid  $\mathcal{C}$ -algebra, with  $\theta_A = 1$ .*

The proof is straightforward.

As in Example 2.14, we also have an action of  $G$  by automorphisms on  $A$  defined by  $\pi_g \delta_h = \delta_{hg^{-1}}$ , and  $A^G = \mathbf{1} = \mathbb{C}(\sum_{h \in G} \delta_h)$ .

We remind the reader that for every  $\mathcal{C}$ -algebra  $A$ , one can define the category  $\text{Rep } A$  of  $A$ -modules and two natural functors  $F: \mathcal{C} \rightarrow \text{Rep } A, G: \text{Rep } A \rightarrow \mathcal{C}$  (see [KO]). The category  $\text{Rep } A$  is a tensor category; we denote tensor product in  $\text{Rep } A$  by  $\otimes_A$ .

**3.2. Theorem.** *The category  $\text{Rep } A$  is equivalent to the category  $G\text{Vec}$  of  $G$ -graded vector spaces. Under this equivalence, the functor  $\otimes_A$  becomes the usual tensor product of vector spaces with the grading given by  $\text{wt}(v \otimes w) = \text{wt}(v)\text{wt}(w)$ , and the functors  $F, G$  are given by*

$$\begin{aligned} F(V) &= V && \text{forgetting the action of } G \text{ but keeping the grading} \\ G(V) &= \mathcal{F}(G) \otimes V && \text{with grading given by } \text{wt}(\delta_g \otimes v) = g\text{wt}(v)g^{-1} \end{aligned}$$

*Proof.* It is immediate from the definition that  $\text{Rep } A$  is the category of  $G$ -modules with  $G \times G$ -grading such that

$$\text{wt}(gv) = (gxg^{-1}, gy) \quad \text{if } \text{wt}(v) = (x, y) \in G \times G$$

Indeed, the action of  $G$  and the first component of the grading define  $V$  as an object of  $\mathcal{C}$ , and the second component of the grading defines the action of  $A$ : if  $v \in V$  is homogeneous, then

$$\delta_g v = \begin{cases} v, & \text{wt}(v) = (\cdot, g) \\ 0, & \text{otherwise} \end{cases}$$

Define a new  $G \times G$ -grading  $\widetilde{\text{wt}}$  by

$$\widetilde{\text{wt}}(v) = (y^{-1}xy, y) \quad \text{if } \text{wt}(v) = (x, y)$$

(which implies that  $\text{wt}(v) = (y\tilde{x}y^{-1}, y)$  if  $\widetilde{\text{wt}}(v) = (\tilde{x}, y)$ ). Then

$$\widetilde{\text{wt}}(gv) = (\tilde{x}, gy) \quad \text{if } \widetilde{\text{wt}}(v) = (\tilde{x}, y).$$

From this it immediately follows that the functor  $\text{Rep } A \rightarrow G\mathcal{V}ec$  given by

$$(3.2) \quad V \mapsto \{v \in V \mid \text{wt}(v) = (\cdot, 1)\} = \{v \in V \mid \widetilde{\text{wt}}(v) = (\cdot, 1)\}$$

considered with  $G$ -grading given by the first component of  $\widetilde{\text{wt}}$ , is an equivalence of categories. The remaining statements of the theorem are straightforward.  $\square$

**3.3. Corollary.** *The set of simple objects in  $\text{Rep } A$  is  $G$ .*

For future use we give description of the corresponding simple objects  $X_g$  in terms of  $G$ -graded vector spaces and in terms of  $G \times G$ -graded  $G$ -modules. As a  $G$ -graded vector space,

$$(X_g)_h = \begin{cases} \mathbb{C}, & g = h \\ 0, & \text{otherwise} \end{cases}$$

As a bi-graded  $G$ -module,  $X_g$  is given by

$$(3.3) \quad X_g = \bigoplus_{y^{-1}xy=g} \mathbb{C}e_{x,y}$$

with  $\text{wt}(e_{x,y}) = (x, y)$ , the action of  $G$  and  $A$  given by

$$\begin{aligned} he_{x,y} &= e_{h x h^{-1}, h y} \\ \mu(\delta_h \otimes e_{x,y}) &= \begin{cases} e_{x,y}, & h = y \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Note that the first component of  $\text{wt}(v)$ ,  $v \in X_g$  is supported on the conjugacy class of  $g$ .

This description immediately implies the following result:

$$(3.4) \quad G(X_g) = V_{g, \mathcal{F}(Z(g))} = \bigoplus_{\pi \in \widehat{Z(g)}} \pi \otimes V_{g, \pi}$$

where  $\pi$  is the representation space, considered with trivial action of  $D(G)$ .

**3.4. Theorem.**  $\text{Rep}^0 A = \mathcal{V}ec$  which is considered as a subcategory in  $G\mathcal{V}ec$  consisting of spaces with grading identically equal to 1.

*Proof.* Let  $V \in \text{Rep } A$ ; for now, we consider  $V$  as a  $G \times G$ -graded  $G$ -module, as in the proof of Theorem 3.2. Then explicit calculation shows that

$$\check{R}_{VA} \check{R}_{AV}(\delta_g \otimes v) = \delta_{xg} \otimes v \quad \text{if } \text{wt}(v) = (x, \cdot).$$

Therefore,

$$\mu_V \check{R}_{VA} \check{R}_{AV}(\delta_g \otimes v) = \begin{cases} v, & xg = y \\ 0, & \text{otherwise} \end{cases}$$

where  $\text{wt}(v) = (x, y)$ . Comparing it with the usual formula for action of  $A$ ,

$$\mu_V(\delta_g \otimes v) = \begin{cases} v, & g = y \\ 0, & \text{otherwise} \end{cases}$$

we see that  $\mu_V = \mu_V \check{R}_{VA} \check{R}_{AV}$  iff  $\text{wt}(v) = (1, \cdot)$  for all  $v \in V$ .  $\square$

For future use, we give here two more results about  $D(G)$ . First, define the map

$$(3.5) \quad \begin{aligned} \tau: D(G) &\rightarrow D(G) \\ g\delta_h &\mapsto g\delta_{h^{-1}} \end{aligned}$$

Then it is trivial to check the following properties.

- 3.5. Lemma.** (1)  $\tau$  is an algebra automorphism.  
 (2)  $\tau$  is coalgebra anti-automorphism:  $\Delta\tau(a) = \tau \otimes \tau(\Delta^{op}a)$ ,  $\tau \circ S = S \circ \tau$ , where  $S$  is the antipode in  $D(G)$ .  
 (3)  $(\tau \otimes \tau)(R) = R^{-1}$ .

Thus, if we define for a representation  $V$  of  $D(G)$  a new representation  $V^\tau$  which coincides with  $V$  as a vector space but with the action of  $D(G)$  twisted by  $\tau$ :  $\pi_{V^\tau}(a) = \pi_V(\tau a)$ , then one has canonical isomorphisms  $(V \otimes W)^\tau = W^\tau \otimes V^\tau$ ,  $(V^*)^\tau = (V^\tau)^*$  and thus,  $\tau$  gives an equivalence

$$(3.6) \quad \tau: (\text{Rep } D(G))^{op} \xrightarrow{\sim} \text{Rep } D(G),$$

where  $(\text{Rep } D(G))^{op}$  coincides with  $\text{Rep } D(G)$  as an abelian category but has tensor product and braiding defined by  $V \otimes^{op} W = W \otimes V$ ,  $R^{op} = R^{-1}$ .

Second, note that  $D(G)$  can be generalized as follows. Let  $H \subset G$  be a normal subgroup. Define

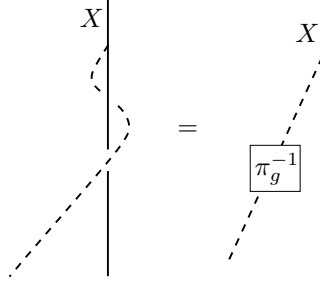
$$(3.7) \quad D(G, H) = \mathbb{C}[G] \ltimes \mathcal{F}(H).$$

One easily sees that it is a quotient of  $D(G)$ :  $D(G, H) = D(G)/J_H$ , where  $J_H$  is the ideal generated by  $\delta_g, g \in G \setminus H$  (this is a Hopf ideal). For  $H = \{1\}$ , we get  $D(G, H) = \mathbb{C}[G]$ ; for  $H = G$ ,  $D(G, H) = D(G)$ . One also easily sees that  $\text{Rep } D(G, H)$  is the subcategory of  $\text{Rep } D(G)$  consisting of representations such that  $\text{wt}(v) \in H$ .

**3.6. Theorem.** Let  $\mathcal{A} \subset \text{Rep } D(G)$  be a full subcategory containing  $\text{Rep } G$  (which we consider as a subcategory in  $\text{Rep } D(G)$  as before), and closed under duality, tensor product, and taking sub-objects. Then  $\mathcal{A} = \text{Rep } D(G, H)$  for some normal subgroup  $H \subset G$ .

*Proof.* The proof is based on the following easily proved lemma.

**3.7. Lemma.** Let  $g_1, g_2 \in G$  and let  $\pi_1, \pi'$  be irreducible representations of  $Z(g_1)$ ,  $Z(g_1 g_2)$  respectively. Then there exists  $\pi_2$  — an irreducible representation of  $Z(g_2)$  such that  $\langle V_{g_1, \pi_1} \otimes V_{g_2, \pi_2}, V_{g_1 g_2, \pi'} \rangle \neq 0$ .

FIGURE 5. Definition of  $g$ -twisted module

Using this lemma with  $g_2 = 1$ , we see that if  $V_{g_1, \pi_1} \in \mathcal{A}$ , for some  $\pi_1 \in \widehat{Z(g_1)}$ , then for all  $\pi \in \widehat{Z(g_1)}$ ,  $V_{g_1, \pi} \in \mathcal{A}$ . Using this lemma again, we see that the set  $H = \{g \in G \mid V_{g, \pi} \in \mathcal{A}\}$  is closed under product. Since  $V_{g, \pi} = V_{hgh^{-1}, \pi \circ h^{-1}}$ , this set is also invariant under conjugation. Thus,  $H$  is a normal subgroup in  $G$ .  $\square$

We also need the following theorem.

**3.8. Theorem.** *Rep  $D(G, H)$  is modular iff  $H = G$ .*

*Proof.* It is well known that  $\text{Rep } D(G)$  is modular; thus, let us prove that if  $H \neq G$  then  $\text{Rep } D(G, H)$  is not modular. As discussed above, simple objects in  $\text{Rep } D(H, G)$  are  $V_{g, \pi}, g \in H$ . Let  $\pi$  be a formal linear combination of irreducible representations of  $G$  such that  $\text{tr}_\pi(h) = 0$  for all  $h \in H$ ; such a  $\pi$  always exists if  $H \neq G$ . Then it follows from explicit formulas for  $s$  (see, e.g., [BK]) that

$$s_{(1, \pi), (h, \pi')} = 0$$

for all  $h \in H, \pi' \in \widehat{Z(h)}$ . Thus,  $s$  is singular.  $\square$

#### 4. TWISTED MODULES

As before, we let  $A$  be a rigid  $\mathcal{C}$ -algebra with  $\theta_A = \text{id}$ , and  $G$  — a compact group acting faithfully on  $G$  by automorphisms (by Corollary 2.12, this implies that  $G$  is finite). For every  $g \in G$ , we will denote the corresponding automorphism of  $A$  by  $\pi_g$ . We use the same notation  $\text{Rep } A, \text{Rep}^0 A, \mu_V: A \otimes V \rightarrow V$  and functors  $F: \mathcal{C} \rightarrow \text{Rep } A, G: \text{Rep } A \rightarrow \mathcal{C}$  as in [KO]. We will also use the same conventions in figures as in [KO], representing  $A$  by dashed line.

From now on, we will also assume that  $A$  is such that  $\text{Rep}^0 A$  has a unique simple module,  $A$  itself; thus,  $\text{Rep}^0 A \simeq \mathcal{V}ec$ . This corresponds to “holomorphic” case in conformal field theory; for this reason, we will call such  $A$  “holomorphic”.

**4.1. Definition.** Let  $g \in G$ . A module  $X \in \text{Rep } A$  is called  $g$ -twisted if

$$\mu \tilde{R}^2 = \mu \circ (\pi_{g^{-1}} \otimes \text{id}): A \otimes V \rightarrow V$$

(see Figure 5).

In particular,  $X$  is 1-twisted iff  $X \in \text{Rep}^0 A$ . This definition is, of course, nothing but rewriting in our language of the definition given in [DVVV].

**4.2. Example.** In the situation of Section 3, the simple module  $X_g$  is  $g$ -twisted. Indeed,

$$\mu \check{R}^2(\delta_h \otimes e_{x,y}) = \begin{cases} e_{x,y}, & xh = y \\ 0, & \text{otherwise} \end{cases}$$

But for  $X_g$ ,  $y^{-1}xy = g$ , which trivially implies that  $xh = y \iff hg = y$ . Thus,  $\mu \check{R}^2(\delta_h \otimes e_{x,y}) = \mu(\delta_{hg} \otimes e_{x,y})$ .

The key result of this section is the following theorem.

**4.3. Theorem.** Assume that  $\text{Rep}^0 A = \mathcal{V}ec$ . Then every simple object  $X_i \in \text{Rep } A$  is  $g$ -twisted for some  $g = g(X_i) \in G$ .

*Proof.* The proof is based on the following lemma.

**4.4. Lemma.** If  $X \in \text{Rep } A$  is simple and  $A$  is holomorphic, then

$$\frac{1}{(\dim A)^2} \text{ (loop with } X^* \text{ and } X \text{)} = \frac{1}{\dim X} \text{ (crossing of } X^* \text{ and } X \text{)}$$

Note that  $\dim X_i$  is non-zero in any semisimple rigid category in which  $X^{**} \simeq X$  (see, e.g., [BK, Section 2.4]). In particular, this implies  $\dim_{\text{Rep } A} X \neq 0$  and thus,  $\dim X = (\dim A)(\dim_{\text{Rep } A} X) \neq 0$  (see [KO, Theorem 3.5]).

*Proof.* Let us rewrite the left hand side as shown in Figure 6. Using Lemma 1.15, Lemma 5.3 from [KO], we see that the left hand side is the composition

$$X^* \otimes X \rightarrow X^* \otimes_A X \xrightarrow{P} (X^* \otimes_A X)^0$$

where  $P$  is the projector  $\text{Rep } A \rightarrow \text{Rep}^0 A$ , and for  $V \in \text{Rep } A$ ,  $V^0 = P(V)$  is the maximal sub-object of  $V$  which lies in  $\text{Rep}^0 A$ . But if  $A$  is holomorphic, then the only simple object in  $\text{Rep}^0 A$  is  $A$  itself, which is the unit object in  $\text{Rep } A$ . It appears in  $X^* \otimes_A X$  with multiplicity one, and the right-hand side is exactly the projection of  $X^* \otimes_A X$  on  $A \subset X^* \otimes_A X$ .

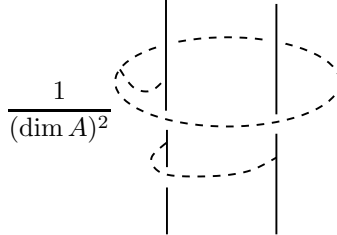


FIGURE 6.

□

(4.1)  $T_X = \mathbf{x} \times \text{elliptical curve}$

*Proof.* Let us calculate  $\mu \circ (T_X \otimes T_X)$ . Using Lemma 4.4 we can rewrite the graph defining  $\mu \circ (T_X \otimes T_X)$  as shown in Figure 7. (Note that we need to use  $\tilde{R}_{AA}^2 = 1$  (Corollary 2.13) to move the “ring” through  $A$ ; the last step also uses Lemma 1.10 from [KO].) This shows that  $\mu \circ (T_X \otimes T_X) = (\dim X)T_X \circ \mu$ . In other words,  $\frac{1}{\dim X}T_X$  is a morphism of  $\mathcal{C}$ -algebras. But it easily follows from Theorem 2.15 that every such morphism is either zero or invertible. Restricting  $T_X$  to  $\mathbf{1} \subset A$ , we see that  $T_X$  is non-zero; thus,  $\frac{1}{\dim X}T_X$  is an automorphism.  $\square$

The statement of the theorem easily follows from these two lemmas. Indeed, combining Lemma 4.5 with Corollary 2.16 we see that  $\frac{1}{\dim X}T_X = \pi_g$  for some  $g \in G$ . On the other hand, Lemma 4.6 gives  $\mu \circ \tilde{R}^{-2} = \mu \circ (\frac{1}{\dim X}T_X \otimes \text{id}) = \mu \circ (\pi_g \otimes \text{id})$  and thus,  $\mu \circ \tilde{R}^2 = \mu \circ (\pi_g^{-1} \otimes \text{id})$ . This completes the proof of Theorem 4.3  $\square$

**4.7. Theorem.** *Let  $X \in \text{Rep } A$  be simple. Then*

- (1) For  $h \in G$ , let  $X^h \in \text{Rep } A$  coincide with  $X$  as an object of  $\mathcal{C}$ , but with a twisted action of  $A: \mu_{X^h} = \mu_X \circ (\pi_h \otimes \text{id})$ . Then  $g(X^h) = h^{-1}g(X)h$ .
- (2)  $g(X) = 1$  iff  $X \simeq A$ .
- (3)  $g(X^*) = g(X)^{-1}$ .
- (4) If  $X, Y \in \text{Rep } A$  are simple then so is  $X \otimes_A Y$  and  $g(X \otimes_A Y) = g(X)g(Y)$ .
- (5) If  $X, Y \in \text{Rep } A$  are simple and non-isomorphic, then  $g(X) \neq g(Y)$ .

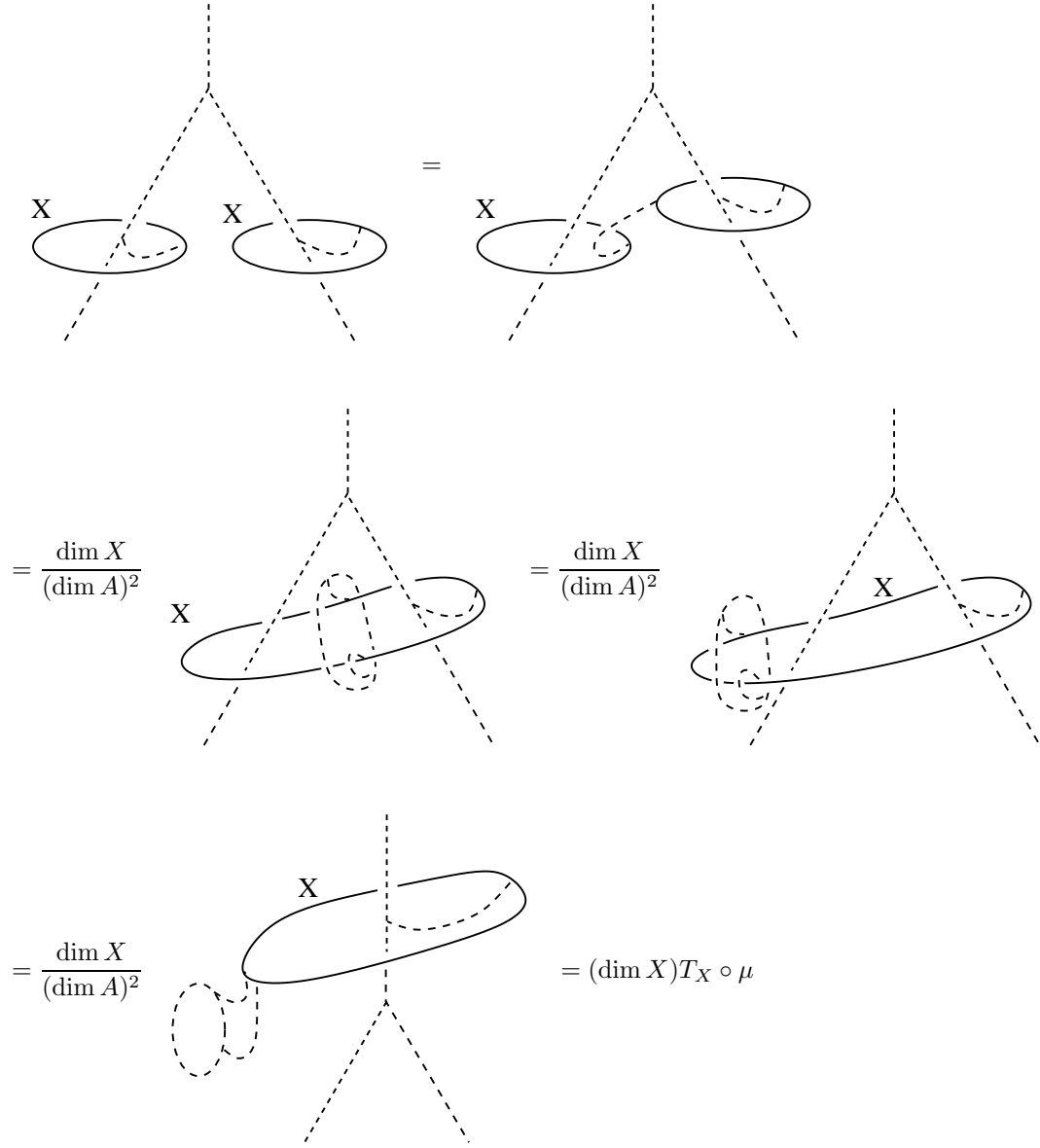


FIGURE 7. Proof of Lemma 4.5

*Proof.* (1) is immediate from the definitions. (2) follows from the fact that the only simple object in  $\text{Rep}^0 A$  is  $A$ . To prove (3) and (4), we will use the following lemma which easily follows from Lemma 1.15 in [KO].

**4.8. Lemma.**  $T_{X \otimes_A Y} = \frac{1}{\dim A} T_X T_Y$ .

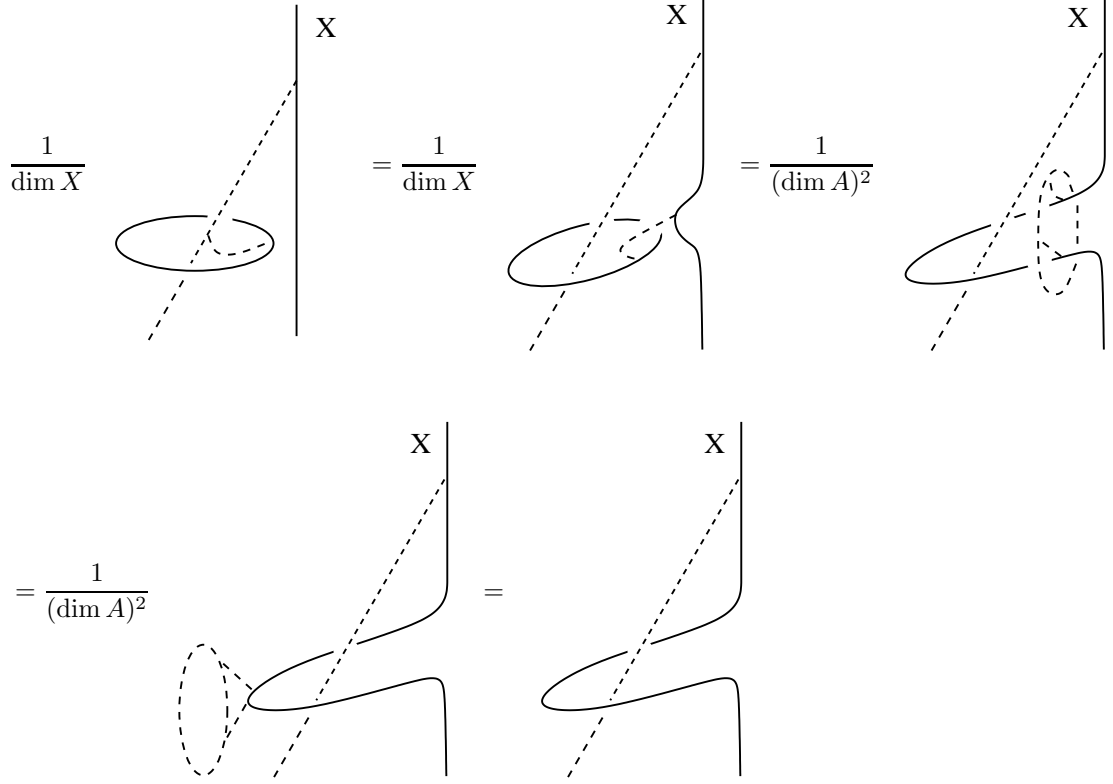


FIGURE 8. Proof of Lemma 4.6

This lemma implies that  $T_{X \otimes_A X^*} = c\pi_{g(X)g(X^*)}$  for some  $c \in \mathbb{C}$ . On the other hand,  $X \otimes_A X^* \simeq A \oplus \sum_{i \neq 0} N_i X_i$ , and  $T_{X \otimes_A X^*} = \pi_1 + \sum c_i \pi_{g(X_i)}$ . Since, by (2),  $g(X_i) \neq 1$  for  $i \neq 0$  and the operators  $\pi_g$  are linearly independent, we see that these two expressions can be equal only if  $X \otimes_A X^* \simeq A$ ,  $g(X)g(X^*) = 1$ .

To prove (4), note that  $X^* \otimes_A (X \otimes_A Y) \simeq (X^* \otimes_A X) \otimes_A Y \simeq A \otimes_A Y \simeq Y$  is simple, which immediately implies that  $X \otimes_A Y$  is simple. The identity  $g(X \otimes_A Y) = g(X)g(Y)$  immediately follows from Lemma 4.8.

Finally, to prove (5) note that (3) and (4) imply  $g(X) = g(Y) \iff g(X^*)g(Y) = g(X^* \otimes_A Y) = 1$ . By (2), this is only possible if  $X^* \otimes_A Y \simeq A$ .  $\square$

**4.9. Corollary.** *The map  $X \mapsto g(X)$  is a bijection between the set of isomorphism classes of simple objects in  $\text{Rep } A$  and some normal subgroup  $H \subset G$ .*

We will denote the unique simple  $g$ -twisted object  $X \in \text{Rep } A$  by  $X_g$ ; then Theorem 4.7 implies that

$$(4.2) \quad X_g \otimes_A X_h \simeq X_{gh}.$$



Combining this with multiplicativity of dimension and [KO, Theorem 3.5], we see that

$$g \mapsto \dim_A X = \frac{\dim X}{\dim A}$$

is a character of the group  $H \subset G$ . Since  $H$  is a finite group, this immediately implies the following result.

**4.10. Lemma.** *For any simple  $X \in \text{Rep } A$ ,  $|\frac{\dim X}{\dim A}| = 1$ .*

In particular, if  $\dim X \geq 0$  (which happens if  $\mathcal{C}$  is a unitary category in the sense of Turaev), then this implies  $\dim X_g = \dim A = |G|$ .

## 5. TWISTED SECTORS

Now that we have a description of irreducible objects in  $\text{Rep } A$  in terms of the group  $G$ , we can move on to our ultimate goal: description of irreducible objects in  $\mathcal{C}$ . As before, we assume that  $A$  is rigid,  $\theta_A = \text{id}$  and holomorphic ( $\text{Rep}^0 A = \mathcal{V}ec$ ), and  $G$  is a finite group acting faithfully on  $A$ .

It is immediate from the identity  $\langle M, G(X) \rangle_{\mathcal{C}} = \langle F(M), X \rangle_{\text{Rep } A}$  (see [KO]) that every simple  $L_i \in \mathcal{C}$  appears in the decomposition of some  $X_g \in \text{Rep } A$  (considered as an object of  $\mathcal{C}$ ). Thus, our first goal is to study the decomposition

$$(5.1) \quad X_g \simeq \bigoplus N_{g,i} L_i.$$

Note that it immediately follows from Theorem 4.7 that as an object of  $\mathcal{C}$ ,  $X_g \simeq X_{hgh^{-1}}$ ; thus, the multiplicities  $N_{g,i}$  only depend on the conjugacy class of  $g$ .

Our strategy in studying decomposition (5.1) is parallel to the approach taken in Section 2 to study the decomposition of  $A = X_1$ . However, instead of the functor  $\Phi: \text{Rep } G \rightarrow \mathcal{C}$  which was defined using  $A$ , we will define functor  $\Phi: \text{Rep } D(G) \rightarrow \mathcal{C}$  using  $\tilde{A} = \bigoplus_{g \in G} X_g$ , where  $D(G)$  is the Drinfeld double of  $G$  (see Section 3).

First of all, we need to define algebra structure on  $\tilde{A}$ . To do so, note that it follows from Theorem 4.7 that for every  $g, h \in G$  there exists a unique up to a constant isomorphism of  $A$ -modules

$$(5.2) \quad \tilde{\mu}_{g,h}: X_g \otimes_A X_h \xrightarrow{\sim} X_{gh}.$$

Considering morphisms  $X_g \otimes_A X_h \otimes_A X_k \rightarrow X_{ghk}$  obtained as compositions of  $\tilde{\mu}$ , we get

$$(5.3) \quad \tilde{\mu}_{g,hk} \circ (1 \otimes_A \tilde{\mu}_{h,k}) = \omega(g, h, k) \tilde{\mu}_{gh,k} \circ (\tilde{\mu}_{g,h} \otimes_A 1)$$

for some  $\omega(g, h, k) \in \mathbb{C}^\times$ . One immediately sees that  $\omega$  is a 3-cocycle on  $G$  with values in  $\mathbb{C}^\times$  and that rescaling  $\tilde{\mu}$  by  $\tilde{\mu}_{g,h} \mapsto \tilde{\mu}_{g,h} \cdot f(g, h)$  results in replacing  $\omega$  by a cohomological one. Thus, the class  $[\omega] \in H^3(G, \mathbb{C}^\times)$  is well-defined.

To simplify the exposition, in this paper we only consider the simplest case  $\omega \equiv 1$ . General case is similar but will involve “twisted” version of Drinfeld double, as in [DPR], and will be discussed elsewhere. Note also that rescaling  $\tilde{\mu}_{g,h} \mapsto c \tilde{\mu}_{g,h}$  where  $c \in \mathbb{C}^\times$  is independent of  $g, h$ , does not change  $\omega$ ; thus, without loss of generality we can assume that  $\tilde{\mu}_{1,1}: A \otimes_A A \rightarrow A$  coincides with the multiplication map  $\mu$ .

**5.1. Assumption.** *From now on, we assume that  $\omega \equiv 1$  and  $\tilde{\mu}_{1,1} = \mu$ .*

In this case, the morphism

$$(5.4) \quad \tilde{\mu} = \bigoplus_{g,h} \tilde{\mu}_{g,h}: \tilde{A} \otimes_A \tilde{A} \rightarrow \tilde{A}$$

is associative. We will also use the the same symbol  $\tilde{\mu}$  for the composition

$$\tilde{A} \otimes \tilde{A} \rightarrow \tilde{A} \otimes_A \tilde{A} \xrightarrow{\tilde{\mu}} \tilde{A}$$

where the first morphism is the canonical projection. This morphism defines on  $\tilde{A}$  the structure of an associative (but not commutative)  $\mathcal{C}$ -algebra.

**5.2. Lemma.** *Under Assumption 5.1,*

- (1) *The morphisms  $\tilde{\mu}_{1,g}: A \otimes X_g \rightarrow X_g, \tilde{\mu}_{g,1}: X_g \otimes A \rightarrow X_g$  coincide with  $\mu_{X_g}, \mu_{X_g} \circ \tilde{R}_{A,X_g}^{-1}$  respectively.*

- (2) *The morphisms*

$$(5.5) \quad \begin{aligned} X_{g^{-1}} \otimes X_g &\xrightarrow{\tilde{\mu}} X_1 = A \rightarrow \mathbf{1} \\ \mathbf{1} \hookrightarrow A = X_1 &\xrightarrow{(\dim A)\tilde{\mu}^{-1}} X_g \otimes_A X_{g^{-1}} \hookrightarrow X_g \otimes X_{g^{-1}} \end{aligned}$$

*satisfy the rigidity axioms and thus define an isomorphism*

$$(5.6) \quad X_{g^{-1}} \simeq X_g^*.$$

*(We use the fact that  $X \otimes_A Y$  is canonically a direct summand in  $X \otimes Y$ , see [KO, Corollary 1.16]).*

- (3) *For all  $g$ ,*

$$\frac{\dim X_g}{\dim A} = 1.$$

The proof is left to the reader as an exercise.

Now let us define an action of  $G$  on  $\tilde{A}$ . It follows from Theorem 4.7 that for every  $g, x \in G$  there exists a unique up to a constant  $A$ -morphism

$$(5.7) \quad \varphi_x(g): X_x^{g^{-1}} \xrightarrow{\sim} X_{gxg^{-1}}.$$

Equivalently,  $\varphi$  is a  $\mathcal{C}$ -morphism  $X_x \rightarrow X_{gxg^{-1}}$  such that

$$\varphi_x(g) \circ \mu \circ (\pi_g^{-1} \otimes 1) = \mu \circ (1 \otimes \varphi_x(g)): A \otimes X_x \rightarrow X_{g^{-1}xg}.$$

Considering composition  $X_x \xrightarrow{\varphi_x(g)} X_{gxg^{-1}} \xrightarrow{\varphi_{gxg^{-1}}(h)} X_{hgx(hg)^{-1}}$  and using uniqueness, we see that

$$(5.8) \quad \varphi_{gxg^{-1}}(h)\varphi_x(g) = c_x(g, h)\varphi_x(hg)$$

for some  $c_x(g, h) \in \mathbb{C}^\times$ .

In particular, denoting by  $Z(x)$  the centralizer of  $x$ :

$$Z(x) = \{g \in G \mid gx = xg\}$$

we see that  $g \mapsto \varphi_x(g)$  defines a projective action of  $Z(x)$  on  $X_x$ .

**5.3. Lemma.** *If  $\omega \equiv 1$ , then  $\varphi_g(x)$  can be chosen so that  $c_x(g, h) \equiv 1$ .*

*Proof.* Define  $\varphi_x(g)$  by Figure 9 (where we used (5.6) to identify  $X_g^* \simeq X_{g^{-1}}$ ). We leave it to the reader to check that so defined  $\varphi$  satisfies the associativity property (5.8) with  $c_x(g, h) \equiv 1$ . □

**5.4. Example.** Let  $x = 1, X_1 = A$ . Then it immediately follows from the construction given in the proof of Theorem 4.3 and Lemma 5.2 that  $\varphi_1(g) = \frac{1}{\dim X_g} T_g = \pi_g$ .

$$\varphi_x(g) = \frac{1}{\dim A} \begin{array}{c} \text{---} \\ | \\ \boxed{\tilde{\mu}} \\ | \\ \text{---} \\ \cup \\ | \\ \text{---} \\ | \\ X_x \end{array} \begin{array}{c} X_g \\ X_g^* \end{array}$$

FIGURE 9. Definition of  $\varphi_x(g): X_x^{g^{-1}} \xrightarrow{\sim} X_{g x g^{-1}}$ 

**5.5. Lemma.** *If  $\omega \equiv 1$  and  $\varphi$  is defined as in Lemma 5.3 then  $\varphi$  is compatible with  $\tilde{\mu}$ , i.e., the following diagram is commutative:*

$$(5.9) \quad \begin{array}{ccc} X_x \otimes X_y & \xrightarrow{\tilde{\mu}} & X_{xy} \\ \varphi_x(g) \otimes \varphi_y(g) \downarrow & & \varphi_{xy}(g) \downarrow \\ X_{g x g^{-1}} \otimes X_{g y g^{-1}} & \xrightarrow{\tilde{\mu}} & X_{g x y g^{-1}} \end{array}$$

**5.6. Remark.** In general case ( $\omega \neq 1$ ), it is easy to see that (5.9) is commutative up to a constant factor  $\gamma_{x,y}(g) \in \mathbb{C}^\times$ . We plan to show in a forthcoming paper that both  $c_x(g, h)$  and  $\gamma_{x,y}(g)$  can be expressed in terms of  $\omega$  in a manner similar to [DPR, Equations 3.5.2, 3.5.3].

Denote

$$\varphi(g) = \bigoplus_x \varphi_x(g): \tilde{A} \rightarrow \tilde{A}.$$

Then we have the following result. (We assume that the reader is familiar with the definition and properties of the Drinfeld double  $D(G)$  of a finite group  $G$ ; these results are briefly reviewed in Section 3.)

**5.7. Theorem.** *Let  $\omega \equiv 1$  and  $\varphi$  defined as in Lemma 5.3. Define the map  $\varphi: D(G) \rightarrow \text{End}_{\mathcal{C}} \tilde{A}$  by*

$$(5.10) \quad \begin{array}{l} g \mapsto \varphi(g) \\ \delta_h \mapsto p_h, \end{array}$$

where  $p_h: \bigoplus X_x \rightarrow \bigoplus X_x$  is the projection on  $X_h$ . Then  $\varphi$  defines an action of  $D(G)$  on  $\tilde{A}$  by  $\mathcal{C}$ -endomorphisms. This action preserves multiplication  $\tilde{\mu}$ : for every  $x \in D(G)$ ,  $\tilde{\mu} \circ (\varphi \otimes \varphi) \Delta(x) = \varphi(x) \circ \tilde{\mu}$ .

*Proof.* Immediately follows from the commutation relations in  $D(G)$ , Lemma 5.3 and Lemma 5.5.  $\square$

Note that it follows from Example 5.4 that restriction of  $\varphi$  to  $\mathbb{C}[G] \subset D(G)$ ,  $A \subset \tilde{A}$  coincides with the original action of  $G$  on  $A$ .

Thus, we have a situation analogous to that of Section 2: we have an associative  $\mathcal{C}$ -algebra  $\tilde{A}$  on which  $D(G)$  acts by endomorphisms. Analogously to definition in Section 1, let  $\mathcal{C}[D(G)]$  be the category with objects: pairs  $(M \in \mathcal{C}, \rho: D(G) \rightarrow$

$\text{End}_{\mathcal{C}}(M)$ ) and morphisms:  $\mathcal{C}$ -morphisms commuting with the action of  $D(G)$ . We have the following results which are parallel to those given in Section 1 for  $\mathcal{C}[G]$ .

**5.8. Lemma.**

- (1)  $\mathcal{C}[D(G)]$  has a canonical structure of a rigid monoidal category
- (2) Both  $\text{Rep } D(G)$  and  $\mathcal{C}$  are naturally subcategories in  $\mathcal{C}[D(G)]$ . This, in particular, allows us to define the functor of exterior tensor product  $\boxtimes: \text{Rep } D(G) \times \mathcal{C}[D(G)] \rightarrow \mathcal{C}[D(G)]$ .
- (3)  $\mathcal{C}[D(G)]$  is a semisimple abelian category with simple objects  $V_{g,\pi} \boxtimes L_i$ .
- (4)  $\mathcal{C}[D(G)]$  is braided with the commutativity isomorphism

$$(5.11) \quad \check{R}^D = \check{R} \circ R^{D(G)}$$

where  $R^{D(G)}$  is the  $R$ -matrix of  $D(G)$ .

- (5) For  $V \in \mathcal{C}, W \in \text{Rep } D[G]$  considered as objects in  $\mathcal{C}[D(G)]$ , one has  $(\check{R}_{VW}^D)^2 = \text{id}$ .

The proof of this lemma is left to the reader as an exercise. Note that unlike  $\mathcal{C}[G]$  case,  $\check{R}^D$  does not coincide with the usual commutativity isomorphism in  $\mathcal{C}$ .

We can also define the notion of “ $D(G)$  invariants”. Namely, define functors

$$\begin{aligned} \text{Rep } D(G) &\rightarrow \mathcal{V}ec \\ V &\mapsto V^{D(G)} = \text{Hom}_{D(G)}(\mathbb{C}, V) \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}[D(G)] &\rightarrow \mathcal{C} \\ \oplus V_i \boxtimes L_i &\mapsto V_i^{D(G)} \boxtimes L_i \end{aligned}$$

(or, more formally, by  $\text{Hom}_{\mathcal{C}}(L, M^{D(G)}) = \text{Hom}_{\mathcal{C}[D(G)]}(L, M)$  where  $L \in \mathcal{C}$  is considered as an object of  $\mathcal{C}[D(G)]$  with trivial action of  $D(G)$ ). Using semisimplicity of  $D(G)$ , one easily sees that  $M^{D(G)}$  is canonically a direct summand in  $M$ , and that for every  $L, M \in \mathcal{C}[D(G)]$  there is a canonical embedding

$$L^{D(G)} \otimes M^{D(G)} \hookrightarrow (L \otimes M)^{D(G)}.$$

**5.9. Theorem.**  $\tilde{A}$  is an associative commutative algebra in  $\mathcal{C}[D(G)]$  (with multiplication  $\tilde{\mu}$  and action of  $D(G)$  defined as in Theorem 5.7), and  $\tilde{A}^{D(G)} = A^G = \mathbf{1}$ .

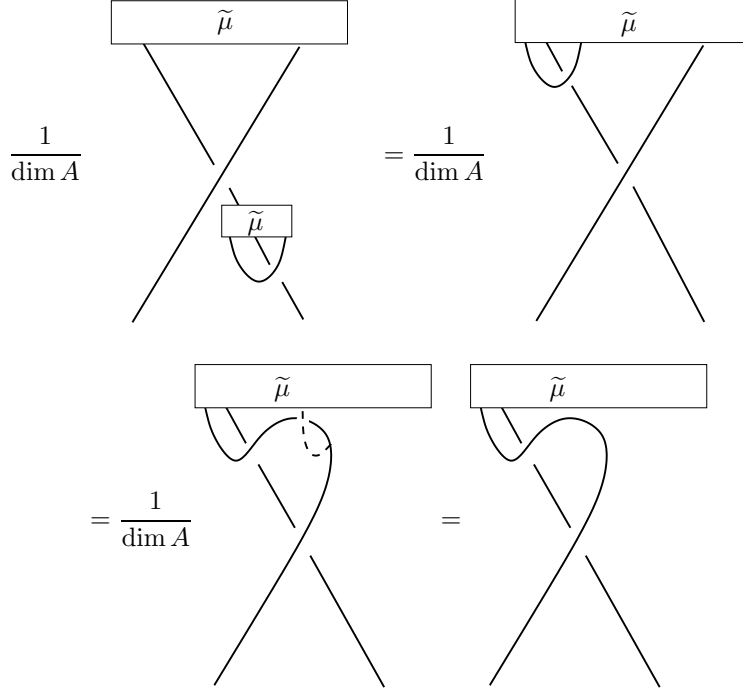
*Proof.* The only part which is not obvious is the fact that  $\tilde{A}$  is commutative (with respect to  $\check{R}^D$ , not  $\check{R}$ !), i.e. that the composition

$$\tilde{A} \otimes \tilde{A} \xrightarrow{R^{D(G)}} \tilde{A} \otimes \tilde{A} \xrightarrow{\check{R}} \tilde{A} \otimes \tilde{A} \xrightarrow{\tilde{\mu}} \tilde{A}$$

coincides with  $\tilde{\mu}$ . To prove it note that explicit formula (3.1) for  $R^{D(G)}$  shows that this composition, when restricted to  $X_{g_1} \otimes X_{g_2}$ , is equal to

$$X_{g_1} \otimes X_{g_2} \xrightarrow{1 \otimes \varphi(g_1)} X_{g_1} \otimes X_{g_1 g_2 g_1^{-1}} \xrightarrow{\check{R}} X_{g_1 g_2 g_1^{-1}} \otimes X_{g_1} \xrightarrow{\tilde{\mu}} X_{g_1 g_2}.$$

Using presentation of  $\varphi(g_1)$  given in Figure 9 and associativity of  $\tilde{\mu}$ , we can rewrite it as shown in Figure 10, which shows that it is equal to  $\tilde{\mu}$ . □

FIGURE 10. Proof of commutativity of  $\tilde{\mu}$  with respect to  $\check{R}^D$ 

Now, define the functor  $\Phi: \text{Rep } D(G) \rightarrow \mathcal{C}$  by

$$(5.12) \quad \Phi(V) = (V \otimes \tilde{A})^{D(G)}$$

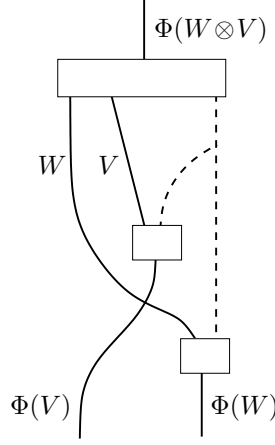
(cf. with (2.1)) and functorial morphism  $J: \Phi(V) \otimes \Phi(W) \rightarrow \Phi(W \otimes V)$  (note that it reverses the order!) by

$$(5.13) \quad \begin{aligned} & (V \otimes \tilde{A})^{D(G)} \otimes (W \otimes \tilde{A})^{D(G)} \xrightarrow{\sim} \left( (V \otimes \tilde{A})^{D(G)} \otimes W \otimes \tilde{A} \right)^{D(G)} \\ & \xrightarrow{\check{R}^D} \left( W \otimes (V \otimes \tilde{A})^{D(G)} \otimes \tilde{A} \right)^{D(G)} \hookrightarrow (W \otimes V \otimes \tilde{A} \otimes \tilde{A})^{D(G)} \\ & \xrightarrow{\tilde{\mu}} (W \otimes V \otimes \tilde{A})^{D(G)} = \Phi(W \otimes V) \end{aligned}$$

(cf. Theorem 2.2). Please note that by Lemma 5.8(5),  $\check{R}^D$  in the second line can be replaced by  $(\check{R}^D)^{-1}$ ; this will be used in the future. The definition of  $J$  is illustrated in Figure 11, where — unlike all previous figures in this paper — crossings represent  $\check{R}^D$  and not  $\check{R}$ , and the dashed line represents object  $\tilde{A}$ . The boxes are the canonical embeddings  $(V \otimes \tilde{A})^{D(G)} \hookrightarrow V \otimes \tilde{A}$  and projections  $V \otimes \tilde{A} \twoheadrightarrow (V \otimes \tilde{A})^{D(G)}$ .

**5.10. Theorem.**  *$J$  is compatible with associativity, unit isomorphisms and reverses commutativity isomorphism, i.e.  $J \circ \check{R}^{-1} = \Phi(\check{R}^{D(G)}) \circ J$ .*

*Proof.* The only one which is not obvious is the commutativity isomorphisms, proof of which is shown in Figure 12, which uses the same conventions as Figure 11.  $\square$

FIGURE 11. Definition of  $J$ 

Now, repeating with obvious changes the same steps as in Section 2, we get the following results. We denote by  $\widehat{D(G)}$  the set of isomorphism classes of irreducible representations of  $D(G)$  and for each  $\lambda \in \widehat{D(G)}$  we choose a representative  $V_\lambda$ .

- 5.11. Theorem.** (1) For any  $\lambda \in \widehat{D(G)}$ ,  $\Phi(V_\lambda)$  is either zero or an irreducible object in  $\mathcal{C}$ .
- (2) If  $\lambda, \mu \in \widehat{D(G)}$  are such that  $\Phi(V_\lambda), \Phi(V_\mu)$  are non-zero, and  $\lambda \neq \mu$ , then  $\Phi(V_\lambda) \not\cong \Phi(V_\mu)$ .
- (3) If  $\lambda \in \widehat{D(G)}$  and  $\Phi(V_\lambda) \neq 0$ , then  $J: \Phi(V_\lambda) \otimes \Phi(V) \rightarrow \Phi(V \otimes V_\lambda)$  is an isomorphism.
- (4) Let  $I = \{\lambda \in \widehat{D(G)} \mid \Phi(V_\lambda) \neq 0\}$ , and let  $\mathcal{A}$  be the abelian subcategory in  $D(G)$  generated by  $V_\lambda, \lambda \in I$ . Then  $\mathcal{A}$  is a subcategory in  $\text{Rep } D(G)$  which is closed under taking submodules, tensor product, and duality.

Combining this last part with Theorem 2.11 and Theorem 3.6, we get the following theorem, which is the main result of this paper.

**5.12. Theorem.** Assume that action of  $G$  is faithful and  $\omega \equiv 1$ . Let  $H \subset G$  be the normal subgroup defined in Corollary 4.9. Then the functor  $\Phi: \text{Rep } D(G) \rightarrow \mathcal{C}$  defined by (5.12) is an equivalence  $(\text{Rep } D(G, H))^{\text{op}} \xrightarrow{\sim} \mathcal{C}$ . (The category  $\text{Rep } D(G, H)$  is defined in (3.7) and  $(\text{Rep } D(G, H))^{\text{op}}$  is  $\text{Rep } D(G, H)$  with opposite tensor product, see (3.6).)

If, in addition,  $\mathcal{C}$  is modular then  $H = G$  and thus,  $\Phi$  is an equivalence  $(\text{Rep } D(G))^{\text{op}} \simeq \mathcal{C}$ .

**5.13. Corollary.** Let  $\tau: \text{Rep } D(G) \rightarrow (\text{Rep } D(G))^{\text{op}}$  be as defined in (3.5). Then the composition  $\Phi \circ \tau: V \mapsto \Phi(V^\tau)$  is an equivalence of tensor categories  $\text{Rep } D(G, H) \xrightarrow{\sim} \mathcal{C}$ . If  $\mathcal{C}$  is modular, this gives an equivalence  $\text{Rep } D(G) \xrightarrow{\sim} \mathcal{C}$ .

Combining this result with the results of [KO], we get the theorem formulated in the introduction.

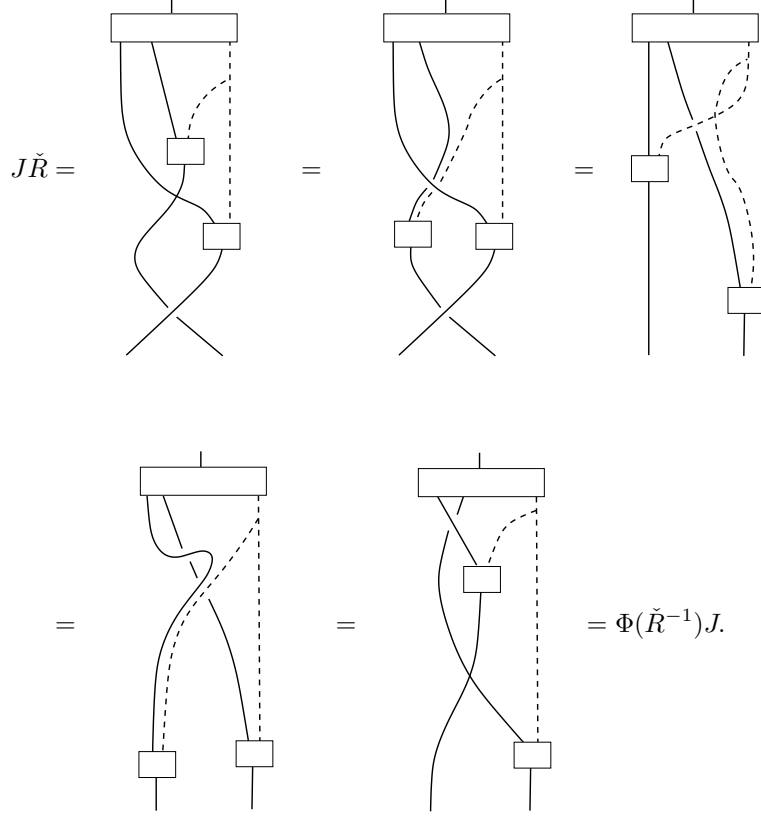


FIGURE 12.

**5.14. Corollary.**

- (1) For every  $g \in H$ ,  $X_g$  considered as an object of  $\mathcal{C}$  has decomposition

$$X_g \simeq \bigoplus_{\pi} \pi \boxtimes \Phi(V_{g,\pi}^*)$$

where the sum is over  $\pi \in \widehat{Z(g)}$ , and  $\pi$  is the vector space of the representation  $\pi$ .

- (2) For every simple object  $L_i \in \mathcal{C}$  there exists a unique conjugacy class  $C$  in  $G$  such that  $L_i$  appears in decomposition of  $X_g$  iff  $g \in C$ .

*Proof.* Follows from the previous corollary and Equation (3.4).  $\square$

**6.  $D(G)$  REVISITED**

It is instructive to explicitly describe the constructions of the previous sections, and in particular equivalence  $\text{Rep } D(G) \xrightarrow{\sim} \mathcal{C}$  in the case when  $\mathcal{C} = \text{Rep } D(G)$ ,  $A = \mathcal{F}(G)$ , so that  $G$  acts on  $A$  by automorphisms and  $\text{Rep}^0 A \xrightarrow{\sim} \text{Vec}$  (see Section 3).

It is natural to expect that in this case, the functor  $\mathcal{C} \xrightarrow{\sim} \text{Rep } D(G)$  is the identity functor; as we will show, this is indeed so.

We already have explicit description of the modules  $X_g$ . The “multiplication” map  $\tilde{\mu}: X_{g_1} \otimes X_{g_2} \rightarrow X_{g_1 g_2}$  is given by

$$e_{x_1, y_1} \otimes e_{x_2, y_2} \mapsto \delta_{y_1, y_2} e_{x_1 x_2, y_1}.$$

We leave it to the reader to check that this map is indeed a morphism of  $A$ -modules, and is associative. Explicit calculation also shows that the map  $\varphi_x(g): X_x \rightarrow X_{gxg^{-1}}$  defined by Figure 9 can be explicitly written as

$$(6.1) \quad \varphi_x(g): e_{a, b} \mapsto e_{a, bg^{-1}}$$

The object

$$\tilde{A} = \bigoplus_g X_g = \bigoplus_{x, y} \mathbb{C} e_{x, y}$$

is a  $D(G) \otimes D(G)$ -module: the first action is defined as in Theorem 5.7 and the second action comes from the fact that each  $X_g$  is an object of  $\mathcal{C} = \text{Rep } D(G)$ . Explicitly, these two actions are written as follows:

$$(6.2) \quad \pi^1(g) e_{x, y} = e_{x, yg^{-1}}, \quad \pi^1(\delta_h) e_{x, y} = \delta_{h, y^{-1}xy} e_{x, y},$$

$$(6.3) \quad \pi^2(g) e_{x, y} = e_{gxg^{-1}, gy}, \quad \pi^2(\delta_h) e_{x, y} = \delta_{h, x} e_{x, y}.$$

This bi-module admits a more explicit description. Namely, consider  $D(G)$  as a  $D(G) \otimes D(G)$ -module by  $\pi^1(x)a = xa, \pi^2(x)a = aS(x)$ , where  $S$  is the antipode in  $D(G)$ . Since  $D(G)$  is semisimple as an associative algebra, we have

$$D(G) \xrightarrow{\sim} \bigoplus V_{g, \pi} \otimes V_{g, \pi}^*$$

considered as a  $D(G) \otimes D(G)$ -module: the first copy of  $D(G)$  acts on the first factor in the tensor product, the second on the second. This also shows that the functor  $V \mapsto (V \otimes D(G))^{D(G)}$  (we consider invariants with respect to the action of  $D(G)$  on the tensor product given by  $\pi^1$ ; thus, this space becomes a  $D(G)$ -module via  $\pi^2$ ) can be canonically identified with the identity functor.

How is this related to  $\tilde{A}$ ? The answer is given by the following simple lemma.

**6.1. Lemma.** *Let  $\tau: D(G) \rightarrow D(G)$  be defined by (3.5), and let  $D(G)^\tau$  be  $D(G)$  considered as a  $D(G) \otimes D(G)$ -module by  $\pi^1(x)a = xa, \pi^2(x)a = aS(\tau x)$ . Then the map*

$$\begin{aligned} \tilde{A} &= \bigoplus \mathbb{C}_{x, y} \rightarrow D(G)^\tau \\ e_{x, y} &\mapsto y^{-1} \delta_x \end{aligned}$$

*is an isomorphism of  $D(G) \otimes D(G)$ -modules.*

The proof is obtained by direct calculation.

Note: the map  $\tilde{\mu}: \tilde{A} \otimes \tilde{A} \rightarrow \tilde{A}$  does **not** coincide with the multiplication in  $D(G)$ !

**6.2. Corollary.** (1) *As a  $D(G) \otimes D(G)$ -module,  $\tilde{A} \simeq V_{g, \pi}^\tau \otimes V_{g, \pi}^*$*   
 (2) *The functor  $\Phi: V \mapsto (V \otimes \tilde{A})^{D(G)}$  is canonically isomorphic to  $\tau$ .*

Thus, the functor  $\Phi \circ \tau$  described in Corollary 5.13 can be identified with the identity functor, as should have been expected.



## REFERENCES

- [BK] Bakalov, B., Kirillov, A., Jr., *Lectures on tensor categories and modular functors*, Amer. Math. Soc., 2000.
- [DLM1] C. Dong, H. Li and G. Mason, *Compact Automorphism Groups of Vertex Operator Algebras*, International Math. Research Notices, **18** (1996), 913-921.
- [DLM2] C. Dong, H. Li and G. Mason, *Regularity of rational vertex operator algebras*, Advances in Math. **132** (1997), 148-166.
- [DLM3] C. Dong, H. Li and G. Mason, *Twisted representations of vertex operator algebras*, Math. Ann. **310** (1998), 571-600.
- [DLM4] C. Dong, H. Li and G. Mason, *Vertex Operator Algebras and Associative Algebras*, J. of Algebra **206** (1998), 67-96.
- [DM1] C. Dong and G. Mason, *Nonabelian Orbifolds and the Boson-Fermion Correspondence*, Comm. Math. Phys. **163** (1994), 523-559.
- [DM2] C. Dong and G. Mason, *On quantum Galois theory*, Duke Math. J. **86** (1997), 305-321.
- [DM3] C. Dong and G. Mason, *Quantum Galois theory for compact Lie groups*, J. Algebra, **214** (1999), 92-102.
- [DM4] C. Dong and G. Mason, *Radical of a vertex operator algebra associated to a module*, arXiv:math.QA/9904155.
- [DM5] C. Dong and G. Mason, *Vertex operator algebras and their automorphism groups*, In: Proceedings of International Conference on Representation Theory (Shanghai, 1998), China Higher Education Press and Springer-Verlag, to appear.
- [DPR] Dijkgraaf, R., Pasquier, V., and Roche, P., *Quasi Hopf algebras, group cohomology and orbifold models*, Nucl. Phys. B (Proc. Suppl.), **18B**(1990), 60-72.
- [DVVV] Dijkgraaf, R., Vafa, C., Verlinde, E., and Verlinde, H., *The operator algebra of orbifold models*, Comm. Math. Phys. **123** (1989), 485-526.
- [DY] Dong, C. and Yamskulna, G. *Vertex operator algebras, generalized doubles and dual pairs*, arXiv:math.QA/0006005
- [FMS] Di Francesco, P., Mathieu, P., and Sénéchal, D., *Conformal field theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 1997.
- [FS] Fuchs, J. and Schweigert, C., *Lie algebra automorphisms in conformal field theory*, arXiv:math.QA/0011160
- [Ka] Kassel, C., *Quantum groups*, Graduate Texts in Mathematics, 155, Springer-Verlag, New York, 1995.
- [KO] Kirillov, A. Jr, Ostrik, V. *On  $q$ -analog of McKay correspondence and ADE classification of  $\widehat{\mathfrak{sl}}_2$  conformal field theories*, arXiv:math.QA/0101219.
- [KT] Kac, V.G. and Todorov, I.T., *Affine orbifolds and rational conformal field theory extensions of  $W_{1+\infty}$* , Comm. Math. Phys. **190** (1997), 57-111.

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